# A production-inventory control model with a mixture of back-orders and lost sales

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# A PRODUCTION-INVENTORY CONTROL MODEL WITH A MIXTURE OF BACK-ORDERS AND LOST-SALES*

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We consider a production-inventory control model of finite capacity, in which backordering up to a certain level is allowed. We assume that there exist two possible production rates. The control is based on two critical stock-levels and prescribes to change the production rate used only when one of these levels is reached. A fixed cost is associated with every switch-over. The rate at which customers arrive and the distribution of the demand of an arriving customer depend on the production rate used at that moment. A formula for the long-run average expected costs per unit time is obtained as a function of the chosen critical levels. From this formula we derive expressions for various interesting operating characteristics of this system, amongst which are the joint stationary distribution of the processes describing the production rate used and the inventory of the system and the average number of switch-overs and lost-sales per unit time.

## 1. Introduction

An important problem in the area of production planning is to dynamically select the production rate in order to cope with the random fluctuation in the demands. Determining how fast the production operation should respond to the demand fluctuations and to what extent these fluctuations should be absorbed by accumulating inventory depends on the relative values of various associated costs. Among these are the production cost rates, inventory cost rate, backorder cost rate, the costs due to the lost sales and the costs of switching production rates. Fixing the production rate at one constant high level causes high production and inventory costs. On the other hand fixing the production rate at a constant low level is associated with high backlogging costs and costs due to lost sales. Because of the random fluctuations in demand the inventory level can become too high or too low even when this constant production rate is selected to match the demand on the average. This obviously results in high associated costs. This suggests dynamic changes in production rates depending on the current inventory level. However, too frequent changes in production rates result in high switch-over costs. Thus a suitable compromise is needed.

Problems of this sort arise in a variety of manufacturing organizations. For example, changes in the through-put rate of chemical processing equipment may be difficult and expensive. The output level of an assembly-line may depend on the number of stations that are manned. The production output of job-shop operation may likewise be influenced by the costs of making changes in the manning level by bringing in new untrained people or laying off people.

The problem of finding a suitable compromise between the mentioned conflicting alternatives has been treated under very special assumptions by Gaver [6] in an initiating study on this subject.

This paper treats a much more general case of this production-inventory control model. We consider the situation in which at any point in time one has to choose one out of *two possible production rates*.

We assume that a switch from one production rate to another takes no time. The rate of the Poisson process according to which customers arrive and the distribution of the successive demands depend on the production rate used. By doing this our

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model contains several interesting special cases like the control of the arrival process or the control of the service process or the control of both. The system is supposed to have a *finite storage capacity*. If the system is at full capacity the production is stopped, until a customer arrives.

Any demand which cannot be fulfilled immediately is backlogged provided the total backlog does not exceed a given level; otherwise it is lost (whether partially or entirely).

The cost structure imposed on this model includes *holding-, shortage- and production costs at a general rate, costs due to lost sales and fixed costs for switching the production rate or restarting production*.

We consider the control rule which is specified by *two critical levels*. This rule prescribes to switch the production rate from fast to slow only when the upper level is reached and to switch from slow to fast only when the inventory has fallen below the bottom level.

For this control rule we derive by a simple approach introduced in Tijms [11] a formula for the long-run average expected costs per unit time. By an appropriate choice of the cost parameters, we may obtain various operating characteristics for the system amongst which are the stationary distribution of the inventory and the average number of switch-overs and lost sales per unit time.

For a special class of cost functions one can use results of Doshi [4] to show that the optimal control rule among all these two-level rules is, in fact, optimal among all reasonable control rules. For the production-inventory model without switch-over costs this was already proved in Doshi [3].

## 2. Mathematical Description of the Model

First we provide the necessary notation. Let $\sigma_1$ and $\sigma_2$ denote the two possible production rates, with $\sigma_1 \geqslant \sigma_2 > 0$. We agree upon saying that the system is in phase $i$ when production rate $\sigma_i$ is used, $i = 1, 2$. Furthermore we define:

$\lambda_i$: = rate of the arrival Poisson process when the system is in phase $i$.

$F_i(\cdot)$: = distribution function of the demand of a customer arriving when the system is in phase $i$.

$U$: = storage capacity of the system.

$L$: = the (nonpositive) level below which the inventory is not allowed to decrease.

$h_i(x)$: = holding, shortage and production cost per unit time when inventory is $x$ and the system is in phase $i$, for $i = 1, 2$ and $x \in [L, U]$.

$p_i(y)$: = penalty cost when an amount $y$ of the demand of a customer, arriving when the system is in phase $i$, is lost.

$\gamma_i$: = fixed cost if production is restarted at level $\sigma_i$.

$\kappa$: = fixed cost for switching the production rate from $\sigma_1$ to $\sigma_2$.

Note that the average expected costs remain the same if there is a switch-over cost $\kappa_1$ for changing from $\sigma_1$ to $\sigma_2$ and $\kappa_2$ for changing from $\sigma_2$ to $\sigma_1$ such that $\kappa_1 + \kappa_2 = \kappa$. We make the following assumptions:

(i) Given that the system is in phase $i$, the demands of arriving customers are independent, $i = 1, 2$.

(ii) $F_i(0) = 0$, $\int_0^\infty x\, dF_i(x) =: \mu_i < \infty$ and $F_i(\cdot)$ is assumed to be continuous, $i = 1, 2$. The continuity of $F_i(\cdot)$ is not essential for our analysis.

(iii) $h_i(\cdot)$ is bounded on $[L, U]$ and has only a finite number of discontinuities for $i = 1, 2$.

(iv) $p_i(\cdot)$ is nondecreasing with $\int_0^\infty p_i(y)\, dF_i(y) < \infty$ for $i = 1, 2$.

The control rule under consideration can be characterized by two levels $y_1$ and $y_2$ such that $L < y_2 \leqslant y_1 < U$. This $(y_1, y_2)$ policy prescribes to switch the production rate from $\sigma_1$ to $\sigma_2$ only when the inventory reaches the value $y_1$ and to switch from $\sigma_2$ to $\sigma_1$ only when the inventory has fallen below level $y_2$. A typical sample path of the process under the $(y_1, y_2)$ control rule is given in Figure 1.

Note that when a customer arrives with a demand $Y$, while the inventory is $x$, the inventory is decreased by an amount of $\min(Y, x - L)$ and an amount of $\max(0, Y - x + L)$ of the excess demand is lost.

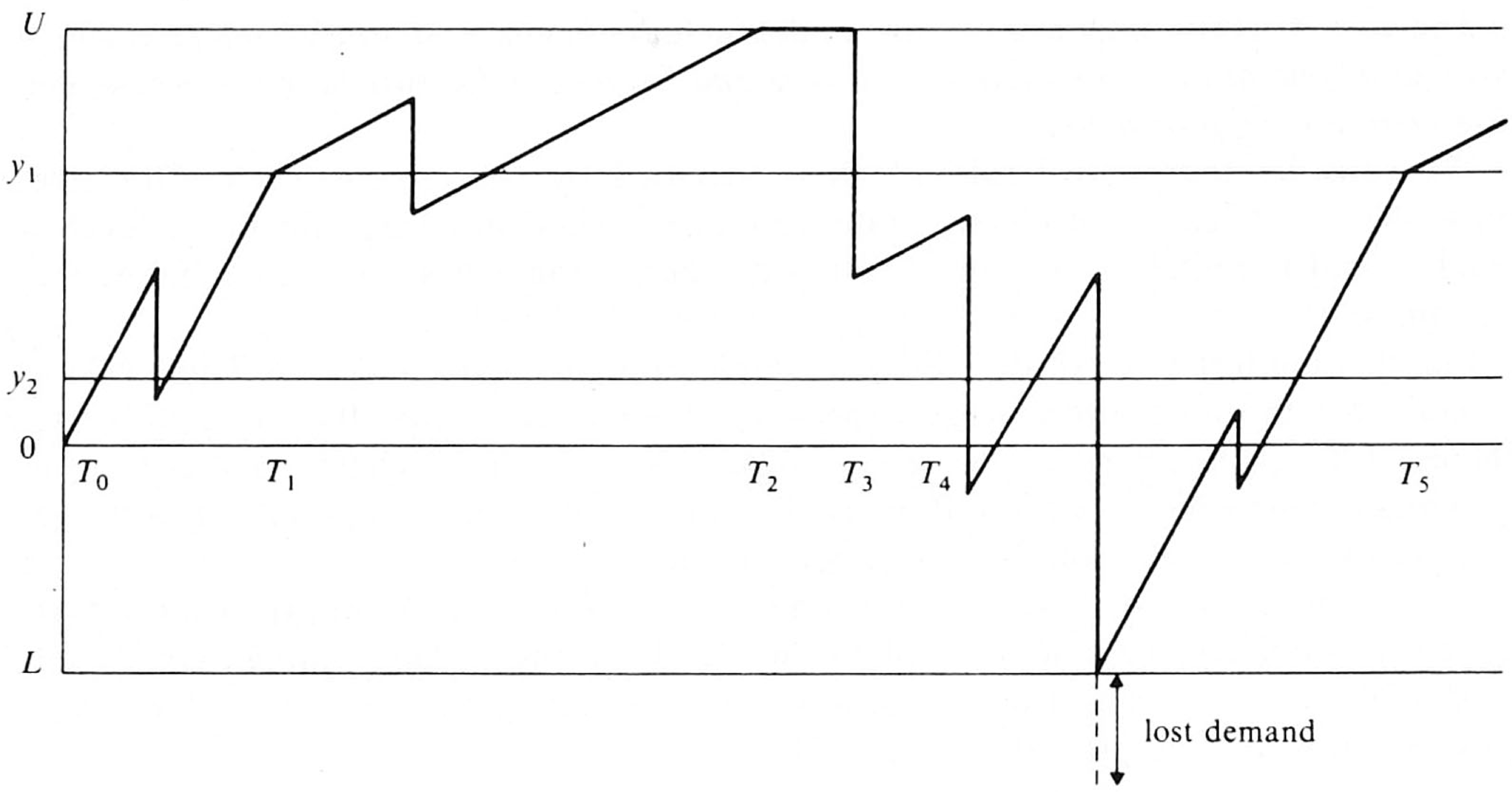


FIGURE 1. High Rate is Used Between $T_0$ and $T_1$ and Between $T_4$ and $T_5$. Low Rate is Used Between $T_1$ and $T_2$ and Between $T_3$ and $T_4$. Production is Stopped Between $T_2$ and $T_3$.

## 3. Evaluation of the $(y_1, y_2)$ Control Rule

Let us define the state of the system as $x$ $(x')$ when the inventory level is $x$ and the system is in phase 1 (2). Denote by $X(t)$ and $S(t)$ the inventory level and the state of the system at time $t$ respectively, where $\{X(t), t \geq 0\}$ and $\{S(t), t \geq 0\}$ are assumed to be continuous from the right. Consider now the system controlled by a fixed $(y_1, y_2)$ control rule. For definiteness let us assume $y_2 > 0$. The case $y_2 \leq 0$ can be treated similarly with obvious modifications in the resulting formulas.

Using a powerful and simple approach based on the analysis of a properly chosen embedded process of $\{S(t), t \geq 0\}$ we shall derive a formula for the long-run average expected costs per unit time of this control rule, cf. also [11] and [12] for other applications of this approach.

Assume the system is empty at time 0. Now let $T_0 = 0$ and let

$T_n$: $=$ $n$th epoch at which either the inventory level reaches $y_1$ while the system is in phase 1 or the inventory falls below $y_2$ while the system is in phase 2 or the inventory level reaches $U$, $n \geq 1$, and

$S_n$: $=$ state of the system at epoch $T_n$, $n \geq 0$.

Then $\{S_n, n \geq 0\}$ is a discrete-time Markov chain embedded in the process $\{S(t), t \geq 0\}$, with state space

$$\mathcal{S} = \{U'\} \cup \{y_1'\} \cup \{x \mid L \leq x < y_2\}.$$

Note that $\{S_n, n = 0, 1, \ldots\}$ never reaches a state in the interval $(y_1', U')$ because inventory increases continuously. Define

$Z(t)$: $=$ total costs incurred in $(0, t]$, $t \geq 0$.

$Z_n$: $=$ total costs incurred in $(T_n, T_{n+1}]$, $n \geq 0$.

$c(s)$: $= E(Z_n \mid S_n = s)$ and $\tau(s)$: $= E(T_{n+1} - T_n \mid S_n = s)$, $s \in \mathcal{S}$.

The proof of the following theorem which is fundamental in our analysis can be found in [11] and [12].

THEOREM 1. *The Markov chain $\{S_n\}$ has a unique invariant probability measure $\pi$ satisfying for any subset $A$ of $\mathbb{S}$*

$$\pi(A) = \int_{\mathbb{S}} q(s, A)\pi(ds) \tag{1}$$

*where $q(\cdot, \cdot)$ denotes the one-step transition probability distribution function of $\{S_n\}$. Moreover*

$$\lim_{t\to\infty} \frac{EZ(t)}{t} = \int_{\mathbb{S}} c(s)\pi(ds) \Big/ \int_{\mathbb{S}} \tau(s)\pi(ds). \tag{2}$$

Next we shall determine the stationary distribution $\pi$ and the functions $c(\cdot)$ and $\tau(\cdot)$. To do this we introduce the following notation. For $i = 1, 2$ and $x \geqslant 0$ let

$$H_i(x) = \frac{\lambda_i}{\sigma_i}\int_0^x (1 - F_i(y))\, dy.$$

Note that $H_i(\infty) = \lambda_i\mu_i/\sigma_i$ denotes the traffic intensity for the system in phase $i$ and that $H_i(\cdot)$ is not a probability distribution function, unless $H_i(\infty) = 1$. By a trick given by Feller [5, pp. 362–363] we shall associate with $H_i(\cdot)$ a probability distribution function that will be involved in the solution of several equations appearing below.

Let $\delta_i$ be defined as the unique root to

$$\int_0^\infty e^{-xy} dH_i(y) = 1, \quad \text{for } i = 1, 2.$$

Then we define for $i = 1, 2$ the probability distribution function $G_i$ by

$$G_i(x) = \int_0^x e^{-\delta_i y} dH_i(y), \quad \text{for } x \geqslant 0.$$

Finally we define the renewal functions $M_i$ by

$$M_i(x) = \sum_{n=1}^{\infty} G_i^{n*}(x), \quad \text{for } x \geqslant 0, \tag{3}$$

where $G_i^{n*}(\cdot)$ is the $n$-fold convolution of $G_i(\cdot)$ with itself. To determine the stationary distribution $\pi$, define for all $x \in [y_2, U]$ and $v \in [L, y_2]$

$p(x, v)$: $=$ probability that the first value of the process $\{S(t), t \geqslant 0\}$ taken on in the set $\{U'\} \cup \{y \mid L \leqslant y < y_2\}$ belongs to the set $\{y \mid L \leqslant y \leqslant v\}$ given $S(0) = x'$, and

$p_0(x)$: $= 1 - p(x, y_2)$, for all $x \in (y_2, U]$.

Observe that $p(\cdot, \cdot)$ is in fact the absorption probability function for a random walk which plays a fundamental role in many queueing and inventory processes (e.g. [1] and [12]).

THEOREM 2. *For any $x \in [y_2, U]$ and $v \in [L, y_2]$*

$$p(x, v) = \phi(x, v) + \int_0^{x-y_2} e^{\delta_2 y}\phi(x - y, v)\, dM_2(y), \tag{4}$$

*where for some constant $c_v$,*

$$\phi(x, v) = c_v - H_2(x - v).$$

*For any $v \in [L, y_2]$ the constant $c_v$ can be determined by the boundary condition $p(U, v) = 0$.*

PROOF. First notice that $p(\cdot, v)$ is a continuous function for all $v \in [L, y_2]$. Suppose the system is in phase 2. The probability of a customer arriving in the time interval of length $\Delta x/\sigma_2$ (in which $\Delta x$ units are produced) is approximately $\lambda_2 \Delta x/\sigma_2$ for small $\Delta x$ and the probability of no arrival during this time interval is approximately $1 - \lambda_2 \Delta x/\sigma_2$. So by appropriate conditioning we have, for $x \in (y_2, U]$

$$p(x - \Delta x, v) = \frac{\lambda_2 \Delta x}{\sigma_2}\left\{1 - F_2(x - v) + \int_0^{x - y_2} p(x - y, v)\, dF_2(y)\right\}$$
$$+ (1 - \lambda_2 \Delta x/\sigma_2) p(x, v) + o(\Delta x),$$

from which we get

$$\frac{\partial p(x, v)}{\partial x} = \frac{\lambda_2}{\sigma_2}\left\{-1 + F_2(x - v) + p(x, v) - \int_0^{x - y_2} p(x - y, v)\, dF_2(y)\right\}. \quad (5)$$

Integration of both sides of (5) yields (cf. [1] and [12]

$$p(x, v) = \phi(x, v) + \int_0^{x - y_2} p(x - y, v)\, dH_2(y), \quad \text{for } x \in [y_2, U]. \quad (6)$$

Equation (6) can be converted with use of $\delta_2$ into a standard renewal equation, the solution of which yields (4) (see [1, p. 77] and [5, p. 362]). Q.E.D.

Write for ease of notation

$$\pi_0 = \pi(\{U'\}), \quad \pi_1 = \pi(\{y'_1\}) \quad \text{and} \quad \pi(v) = \pi\{y \mid L \leqslant y \leqslant v\}, \quad \text{for } v \in [L, y_2].$$

THEOREM 3. *The stationary distribution* $\pi$ *is given by*

$$\pi_0 = p_0(y_1)\left\{2 - 2\int_0^{U - y_2} p_0(U - y)\, dF_2(y) + p_0(y_1)\right\}^{-1},$$

$$\pi_1 = \tfrac{1}{2}(1 - \pi_0),$$

$$\pi(v) = \pi_0\left\{1 - F_2(U - v) + \int_0^{U - y_2} p(U - y, v)\, dF_2(y)\right\} + \tfrac{1}{2}(1 - \pi_0) p(y_1, v),$$

$$\textit{for } v \in [L, y_2].$$

PROOF. The proof of this theorem is immediately from relation (1), the definition of $p(\cdot, \cdot)$ and the relation $\pi_0 + \pi_1 + \pi(y_2) = 1$. Q.E.D.

To determine the functions $c(\cdot)$ and $\tau(\cdot)$ we define

$k_1(x)$: = the expected holding (shortage) and penalty costs incurred up to the first epoch at which the process $\{S(t), t \geqslant 0\}$ reaches the state $y_1$ given that $S(0) = x$, $L \leqslant x \leqslant y_1$, and

$k_2(x)$: = the expected holding (shortage) and penalty costs incurred up to the first epoch at which the process $\{S(t), t \geqslant 0\}$ reaches either $U'$ or the set $\{y \mid L \leqslant y < y_2\}$ given that $S(0) = x'$, $y_2 \leqslant x \leqslant U$.

From the definition of the function $c(\cdot)$ we have the following theorem.

THEOREM 4.

$$c(U) = \frac{h_2(U)}{\lambda} + \int_0^{U - y_2}\{\gamma_2 + k_2(U - y)\}\, dF_2(y) + \int_{U - L}^{\infty} p_2(y - U + L)\, dF_2(y)$$
$$+ \gamma_1(1 - F_2(U - y_2)).$$

$$c(v) = \kappa + k_1(v), \quad \textit{for } v \in [L, y_2].$$

$$c(y_1) = k_2(y_1).$$

The formula for $\tau(s)$ follows from the corresponding one for $c(s)$ by putting $h_i(x) = 1$ for $x \in [L, U]$, $p_i(y) = 0$ for $y \geqslant 0$, $i = 1, 2$ and $\gamma_1 = \gamma_2 = \kappa = 0$.

THEOREM 5.

$$k_1(x) = b_1 - d_1(x) + \int_0^{x-L} \{b_1 - d_1(x-y)\} e^{\delta_1 y}\, dM_1(y) \tag{7}$$

*where*

$$d_1(x) = \int_L^x \left\{ \frac{h_1(z)}{\sigma_1} + \frac{\lambda_1}{\sigma_1} \int_{z-L}^{\infty} p_1(y - z + L)\, dF_1(y) \right\} dz + b_1 H_1(x - L),$$

$$L \leqslant x \leqslant y_1.$$

$$k_2(x) = b_2 - d_2(x) + \int_0^{x-y_2} \{b_2 - d_2(x-y)\} e^{\delta_2 y}\, dM_2(y) \tag{8}$$

*where*

$$d_2(x) = \int_{y_2}^x \left\{ \frac{h_2(z)}{\sigma_2} + \frac{\lambda_2}{\sigma_2} \int_{z-L}^{\infty} p_2(y - z + L)\, dF_2(y) \right\} dz, \qquad y_2 \leqslant x \leqslant U.$$

*The constants $b_1$ and $b_2$ are determined by the boundary conditions*

$$\lim_{x \uparrow y_1} k_1(x) = 0 \quad \textit{and} \quad \lim_{x \uparrow U} k_2(x) = 0$$

PROOF. The proof of this theorem is quite similar to that of Theorem 2. First note that both $k_1(\cdot)$ and $k_2(\cdot)$ are continuous as is immediately clear. For all $x \in (L, y_1]$ for which $h_1(\cdot)$ is continuous at $x$ it follows by the same conditioning arguments as in the proof of Theorem 2 that

$$k_1(x - \Delta x) = \frac{h_1(x)\Delta x}{\sigma_1} + \frac{\lambda_1 \Delta x}{\sigma_1} \left\{ \int_0^{x-L} k_1(x-y) dF_1(y) \right.$$
$$\left. + \int_{x-L}^{\infty} \{k_1(L) + p_1(y - x + L)\} dF_1(y) \right\} + \left(1 - \frac{\lambda_1 \Delta x}{\sigma_1}\right) k_1(x) + o(\Delta x).$$

From this equation we get for all $x \in [L, y_1]$ at which $h_1(\cdot)$ is continuous

$$k'_1(x) = -\frac{h_1(x)}{\sigma_1} + \frac{\lambda_1}{\sigma_1} k_1(x) - \frac{\lambda_1}{\sigma_1} \int_0^{x-L} k_1(x-y) dF_1(y)$$
$$- \frac{\lambda_1}{\sigma_1} \int_{x-L}^{\infty} \{k_1(L) + p_1(y - x + L)\} dF_1(y).$$

Integration of both sides yields an equation which can again be converted into a renewal equation. The solution of this equation, together with

$$k_1(L) = b_1 - d_1(L) = b_1,$$

gives (7) (see also [12]). The proof of (8) is quite similar. Q.E.D.

We now have completed the calculation of $\pi$, $c(\cdot)$ and $\tau(\cdot)$ and so by equation (2) we have a formula for the average cost of the $(y_1, y_2)$ control rule. Using this formula and making special choices for the cost functions we can deduce several important operating characteristics of the system. To obtain the stationary distribution of the inventory define for any $t \geqslant 0$ the random variable $A(t)$ denoting the phase of the system at time $t$, where the process $\{A(t), t \geqslant 0\}$ is continuous from the right.

Now fix $k \in \{1, 2\}$ and $z \in [L, U]$. We have (e.g. page 99 in [9])

$$\lim_{t\to\infty} \Pr(A(t) = k;\ X(t) \leqslant z)$$
$$= \text{long-run expected fraction of time the system is in phase-}k\text{ and the inventory is less than or equal to } z. \tag{9}$$

However, by choosing $h_k(x) = 1$ for $x \in [L, z]$, $h_k(x) = 0$ for $x \in (z, U]$ and taking the other cost functions and parameters equal to zero the right hand side of (9) is represented by $\lim_{t\to\infty}(EZ(t)/t)$. Hence for this special choice of the cost functions and parameters $\lim_{t\to\infty}(EZ(t)/t)$ gives the joint stationary distribution of the inventory and the phase of the system.

Similarly by choosing $p_k(y) = 1$ for $y > z$ and $p_k(y) = 0$ for $y \in [0, z]$ for some $k \in \{1, 2\}$ and $z > 0$ and choosing all other cost functions and parameters equal to zero $\lim_{t\to\infty}(EZ(t)/t)$ represents the average number of lost sales larger than $z$ per unit time. Finally putting all cost functions equal to zero and $\gamma_i = 1$ for fixed $i$ (resp. $\kappa = 1$) $\lim_{t\to\infty}(EZ(t)/t)$ represents the average number of times production is restarted at rate $\sigma_i$ per unit time (resp. the average number of times the production rate is switched from $\sigma_1$ to $\sigma_2$ per unit time).

From these results it is clear that for any $(y_1, y_2)$ policy we may obtain numerical results for all mentioned operating characteristics once we have determined the renewal functions $M_i(\cdot)$, $i = 1, 2$ defined by (3). From renewal theory we know that $M_i(\cdot)$ is the unique function which is bounded on finite intervals and satisfies for $x \geqslant 0$

$$M_i(x) = G_i(x) + \int_0^x M_i(x - y) g_i(y)\, dy, \tag{10}$$

where $g_i(\cdot)$ is the probability density function of $G_i(\cdot)$. From (10) the function $M_i(\cdot)$ can be numerically solved (cf. [7]) or alternatively approximating formulas for $M_i(\cdot)$ can be used (see [8] or p. 357 in [5]).

Finally we note that for a production-inventory control problem with no backlogging, a finite number of production rates and a linear cost structure Weeda [13] has developed an efficient algorithm based on a general Markov decision method (cf. [2]) to compute the optimal switch-over levels.

## 4. The Exponential Case

In this section we give some explicit results concerning a case in which the renewal functions $M_i(\cdot)$ can be explicitly determined. Let for $x \geqslant 0$ and $i = 1, 2$,

$$F_i(x) = 1 - \exp(-\eta_i x).$$

In this case the renewal function is linear. We find $\delta_i = -\eta_i + \lambda_i/\sigma_i$ and $M_i(y) = \lambda_i y/\sigma_i$. In the following formulas it is assumed that $\lambda_i \neq \sigma_i\eta_i$ for $i = 1, 2$. Put for abbreviation $\alpha_i = \lambda_i/\sigma_i$, $\beta_i = \eta_i - \alpha_i$, for $i = 1, 2$ and

$$R(y_1, y_2) = \beta_2^{-1}\{\eta_2\exp\{\beta_2(U - y_2)\} - \alpha_2\exp\{\beta_2(U - y_1)\}\}.$$

Then

$$\pi_0 = \frac{R(y_1, y_2)}{2 + R(y_1, y_2)}, \qquad \pi_1 = \frac{1}{2 + R(y_1, y_2)},$$

$$\pi(v) = \frac{\exp\{-\eta_2(y_2 - v)\}}{2 + R(y_1, y_2)} \qquad \text{for } v \in [L, y_2].$$

Denoting

$$D(y_1, y_2) = \left(\frac{\eta_1}{\sigma_1\beta_1} - \frac{\eta_2}{\sigma_2\beta_2}\right)\left(y_1 - y_2 + \frac{1}{\eta_2}\right) + \frac{\eta_2}{\lambda_2\beta_2} R(y_1, y_2)$$
$$- \frac{\eta_1}{\eta_2\sigma_1\beta_1}\exp\{-\eta_2(y_2 - L)\}$$
$$+ \frac{\alpha_1\eta_2}{\sigma_1\beta_1^2(\beta_1 - \eta_2)}\{\exp\{-\beta_1(y_2 - L)\} - \exp\{-\eta_2(y_2 - L)\}\}$$
$$+ \frac{\alpha_1}{\sigma_1\beta_1^2}\{\exp\{-\beta_1(y_1 - L)\} - \exp\{-\eta_2(y_2 - L)\}\}$$

we find

$$\lim_{t\to\infty} \Pr(A(t) = 1;\ X(t) \leqslant z)$$
$$= \frac{1}{D(y_1, y_2)}\left[\frac{\eta_1 - \eta_2}{\eta_2\sigma_1(\beta_1 - \eta_2)}\{\exp\{-\eta_2(y_2 - z)\} - \exp\{-\eta_2(y_2 - L)\}\}\right.$$
$$+ \frac{\alpha_1}{\sigma_1\beta_1^2}\{\exp\{-\beta_1(y_1 - L)\} - \exp\{-\beta_1(y_1 - z)\}\}$$
$$\left.+ \frac{\alpha_1\eta_2}{\sigma_1\beta_1^2(\beta_1 - \eta_2)}\{\exp\{-\beta_1(y_2 - L)\} - \exp\{-\beta_1(y_2 - z)\}\}\right],$$
$$L \leqslant z \leqslant y_2.$$

$$\lim_{t\to\infty} \Pr(A(t) = 1;\ X(t) \leqslant z)$$
$$= \frac{1}{D(y_1, y_2)}\left[\frac{\eta_1 - \eta_2}{\eta_2\sigma_1(\beta_1 - \eta_2)}\{1 - \exp\{-\eta_2(y_2 - L)\}\}\right.$$
$$+ \frac{\alpha_1}{\sigma_1\beta_1^2}\{\exp\{-\beta_1(y_1 - L)\}$$
$$- \exp\{-\beta_1(y_1 - z)\}\} + \frac{\eta_1}{\sigma_1\beta_1}(z - y_2)$$
$$\left.+ \frac{\alpha_1\eta_2}{\sigma_1\beta_1^2(\beta_1 - \eta_2)}\{\exp\{-\beta_1(y_2 - L)\} - 1\}\right],$$
$$y_2 \leqslant z \leqslant y_1.$$

$$\lim_{t\to\infty} \Pr(A(t) = 2;\ X(t) \leqslant z)$$
$$= \frac{1}{D(y_1, y_2)}\left[\frac{\eta_2}{\sigma_2\beta_2^2}\{\exp\{-\beta_2(y_2 - z)\} - 1\} - \frac{\eta_2}{\sigma_2\beta_2}(z - y_2)\right],$$
$$y_2 \leqslant z \leqslant y_1.$$

$$\lim_{t\to\infty} \Pr(A(t) = 2;\ X(t) \leqslant z)$$
$$= \frac{1}{D(y_1, y_2)}\left[\frac{1}{\sigma_2\beta_2^2}\{\eta_2\exp\{-\beta_2(y_2 - z)\} - \alpha_2\exp\{-\beta_2(y_1 - z)\}\}\right.$$
$$\left.- \frac{\eta_2}{\sigma_2\beta_2}\left(y_1 - y_2 + \frac{1}{\eta_2}\right)\right], \qquad y_1 \leqslant z < U.$$

and

$$\lim_{t\to\infty} \Pr(A(t)=2;\ X(t)\leqslant U)$$

$$= \frac{1}{D(y_1,y_2)}\left[\frac{\eta_2}{\lambda_2\beta_2}R(y_1,y_2)-\frac{\eta_2}{\sigma_2\beta_2}\left(y_1-y_2+\frac{1}{\eta_2}\right)\right].$$

Also we find:

average number of switch-overs from $\sigma_1$ to $\sigma_2$ per unit time $= 1/D(y_1,y_2)$;

average number of times production is restarted at rate $\sigma_1$ per unit time

$$=\frac{R(y_1,y_2)}{D(y_1,y_2)}\exp\{-\eta_2(U-y_2)\};$$

average number of times production is restarted at rate $\sigma_2$ per unit time

$$=\frac{R(y_1,y_2)}{D(y_1,y_2)}(1-\exp\{-\eta_2(U-y_2)\});$$

average number of lost sales which are larger than $z$ and occur while the system is in phase 1 per unit time

$$=\frac{\exp(-\eta_1 z)}{D(y_1,y_2)}\left[\frac{\alpha_1\eta_2}{\beta_1(\beta_1-\eta_2)}\{\exp\{-\eta_2(y_2-L)\}-\exp\{-\beta_1(y_2-L)\}\}\right.$$
$$\left.+\frac{\alpha_1}{\beta_1}\{\exp\{-\eta_2(y_2-L)\}-\exp\{-\beta_1(y_1-L)\}\}\right];$$

average number of lost sales which are larger than $z$ and occur while the system is in phase 2 per unit time

$$=\frac{1}{D(y_1,y_2)}\exp\{-\eta_2(z+y_2-L)\}.^1$$

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